THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H/I/J University Mathematics 2017-2018
Assignment 7
Due Date: 25 Apr 2018 (Wed)

1. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$

   (b) $\displaystyle\int \frac{x^3-3x-2}{x^2+x}\,dx$

   (c) $\displaystyle\int \frac{3x+2}{x^3-1}\,dx$

   (d) $\displaystyle\int \frac{6x+11}{(x+1)^2}\,dx$

2. (a) Prove that $\displaystyle\int_0^1 \frac{u^4(1-u)^4}{1+u^2}\,du = \frac{22}{7}-\pi$.

   (b) Evaluate $\displaystyle\int_0^1 u^4(1-u)^4\,du$ and hence show that

$$\frac{22}{7}-\frac{1}{630} < \pi < \frac{22}{7}-\frac{1}{1260}.$$

3. (a) Let $f(x)$ be an increasing function. Show that

$$\sum_{i=1}^{n-1} f(i) \le \int_1^n f(x)\,dx \le \sum_{i=2}^{n} f(i)$$

   for $n = 2, 3, 4, \cdots$.

   (b) Hence, prove that

$$\ln[(n-1)!] \le \int_1^n \ln x\,dx \le \ln(n!)$$

   and that

$$(n-1)! \le n^n e^{-n+1} \le n!.$$

   By using the result in (a) and that $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$, evaluate $\lim\limits_{n\to\infty} \dfrac{(n!)^{\frac{1}{n}}}{n}$.

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function (i.e. $f'(x)$ is continuous) and let $p \le 1$. Prove that for all $x \in \mathbb{R}$,

$$\int_0^x (x-t)^p f'(t)\,dt = -x^p f(0) + p\int_0^x f(t)(x-t)^{p-1}\,dt.$$

   (b) For any positive integer n and real number x, show that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}e^t\,dt.$$

   Hence, show that

$$\left|\left(e+\frac{1}{e}\right) - 2\left(1+\frac{1}{2!}+\frac{1}{4!}+\cdots+\frac{1}{(2n)!}\right)\right| < \frac{3}{(2n)!}.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with continuous second derivative (i.e. $f''(x)$ exists and it is continuous) and define $I = \int_0^1 f(x)\,dx$.

(a) Show that

$$I = f(0) + \int_0^1 (1-x)f'(x)\,dx = f(1) - \int_0^1 xf'(x)\,dx.$$

Hence, deduce that

$$I = \frac{f(0)+f(1)}{2} - \frac{1}{2}\int_0^1 x(1-x)f''(x)\,dx.$$

(b) Suppose that for all $x \in [0,1]$, there exists constants M and K such that

$$|f'(x)| \le M \qquad \text{and} \qquad |f''(x)| \le K.$$

Show that $\left| I - \dfrac{f(0)+f(1)}{2} \right|$ is bounded above by $\min\{\frac{M}{4}, \frac{K}{12}\}$.

(Remark: What is the geometrical meaning of the quantity $\dfrac{f(0)+f(1)}{2}$?)

6. For any nonnegative integer n, define $I_n = \int_0^{\pi/2} \sin^{2n+1} x\,dx$.

(a) (i) Evaluate I_0 and express I_n in terms of I_{n-1} for any positive integer n.

(ii) Show by mathematical induction that for $n = 0, 1, 2, \cdots$, $I_n = \dfrac{(n!)^2 2^{2n}}{(2n+1)!}$.

(b) For any nonnegative integer n, define $S_m = \sum_{n=0}^{m} \dfrac{(n!)^2 2^{n+1}}{(2n+1)!}$.

(i) Show that

$$S_m = \int_0^{\pi/2} 2\sin x \frac{1 - \left(\frac{1}{2}\sin^2 x\right)^{m+1}}{1 - \frac{1}{2}\sin^2 x}\,dx.$$

(ii) Deduce that

$$\int_0^{\pi/2} \frac{2\sin x}{1 - \frac{1}{2}\sin^2 x}\,dx - \frac{\pi}{2^m} \le S_m \le \int_0^{\pi/2} \frac{2\sin x}{1 - \frac{1}{2}\sin^2 x}\,dx.$$

Hence, show that $\sum_{n=0}^{\infty} \dfrac{(n!)^2 2^{n+1}}{(2n+1)!} = \pi$

7. Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n t\,dt$, where n is a nonnegative integer.

(a) (i) Evaluate I_0 and I_1.

(ii) Show that $I_{n+2} = \dfrac{n+1}{n+2} I_n$ for $n \ge 0$.
Hence, evaluate I_{2m} and I_{2m+1} for $m \ge 1$.

(b) Show that $I_{2m-1} \ge I_{2m} \ge I_{2m+1}$ for $m \ge 1$.

(c) Let $A_n = \dfrac{1}{2n+1}\left[\dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right]^2$, where n is a nonnegative integer.

(i) Using (a) and (b), prove that $\dfrac{2n+1}{2n} A_n \ge \dfrac{\pi}{2} \ge A_n$.

(ii) Show that $\{A_n\}$ is a monotonic increasing sequence.

(iii) Evaluate $\lim_{n\to\infty} \dfrac{1}{\sqrt{2n+1}}\left[\dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right]$.